The Free and Forced Vibrations of a Closed Elastic Spherical Shell Fixed to an Equatorial Beam—Part II: Perturbation Approximations

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The two small parameters that appear in the final equations developed in Part I (Simmonds and Hosseinbor, 2010, "The Free and Forced Vibrations of a Closed Elastic Spherical Shell Fixed to an Equatorial Beam—Part I: The Governing Equations and Special Solutions," ASME J. Appl. Mech., 77, p. 021017), namely, h/R, the ratio of the constant shell thickness to the radius of curvature of the shell's reference surface and H/R, where H is the depth (or width) of the equatorial beam, are exploited using perturbation techniques (including the WKB method). The natural frequencies depend not only on these parameters, but also on the ratio of the mass densities of the shell and beam, the ratio of the Young's moduli of the shell and beam, Poisson's ratio, and the circumferential wave number m. Short tables for typical parameter values are given for those cases where the frequency equation is not explicit. [DOI: 10.1115/1.3197211]

1 Analysis

(In what follows, equation numbers in Ref. [1] will be preceded by the Roman numeral I.) The simplest manifestation of the effect of the small parameter ε^2 is in the expanded form of Eq. (I.76),

$$-\varepsilon^2\lambda^3+\varepsilon^2[(1-\nu)a+3+\mu]\lambda^2+[\Omega^2-1-\underline{\varepsilon^2(3+\mu)(1-\nu)a}]\lambda + b+\underline{2\varepsilon^2(1+\mu)(1-\nu)a}=0 \tag{1}$$

where the error in neglecting the underlined terms is less than that in classical shell theory and

$$a \triangleq 1+(1+\nu)\Omega^2, \quad b\triangleq a[2-(1-\nu)\Omega^2] \tag{2}$$

A standard perturbation analysis of the roots of this cubic yields three cases. Thus, let $\lambda=\varepsilon^{-p}y(\varepsilon)$, $p>0$, $y(0)\neq 0$. Then, with $\Omega^2=1+c\varepsilon^{\alpha}$ and c nonzero and independent of ε, we have the following.

For Case (i), $0\leq\alpha<2/3$,

$$\lambda_1=-\varepsilon^{-\alpha}[b/c+\mathrm{O}(\varepsilon^{2-3\alpha})], \quad \begin{Bmatrix}\lambda_2\\ \lambda_3\end{Bmatrix}=\varepsilon^{(1/2)\alpha-1}[\mp\sqrt{c}+\mathrm{O}(\varepsilon^{1-3\alpha/2})] \tag{3}$$

For Case (ii), $\alpha=2/3$,

$$\lambda_1=\varepsilon^{-2/3}[X+Y+\mathrm{O}(\varepsilon^{2/3})],$$

$$\{\lambda_2,\lambda_3\}=\tfrac{1}{2}\varepsilon^{-2/3}[-(Y+X)\pm i\sqrt{3}(X-Y)+\mathrm{O}(\varepsilon^{2/3})] \tag{4}$$

where

$$X\triangleq\sqrt[3]{\frac{1}{2}b+\sqrt{d}}, \quad Y\triangleq\sqrt[3]{\frac{1}{2}b-\sqrt{d}}, \quad d\triangleq\frac{b^2}{4}-\frac{c^3}{27}, \quad c<3 \tag{5}$$

For Case (iii), $2/3<\alpha$,

$$\{\lambda_1,\lambda_2,\lambda_3\}=\varepsilon^{-2/3}[\sqrt[3]{b}\{1,e^{i2\pi/3},e^{-i2\pi/3}\}+\mathrm{O}(\varepsilon^{\alpha-2/3})] \tag{6}$$

Note that in Cases (ii) and (iii), we may take $b=(2+\nu)(1+\nu)$.

In a strictly rigorous analysis, the next step would be to seek asymptotic expansions of the Legendre functions for large λ. However, it is simpler and more edifying to begin with a two-scale analysis of Eqs. (I.67) and (I.68). Moreover, because we expect Ω^2 to be a continuous function of ε in the narrow *transition range* $|\Omega^2-1|=\mathrm{O}(\varepsilon^{\alpha})$, $\alpha>0$ and because we shall henceforth concentrate on the *resonant frequencies* of the beam-shell configuration—these are the most important features for applications—we shall forego a detailed perturbation analysis of solutions in the transition region and consider only *low*, $0<\Omega^2\leq 1-|c|$, and *high*, $\Omega^2\geq 1+|c|$, dimensionless frequencies.

Ross, in Ref. [2], carried out an asymptotic analysis of the axisymmetric vibrations of a nonshallow spherical cap subject to a variety of boundary conditions along a circular edge. As we do, Ross avoids considering the transition region. However, Ross erroneously states that "Near $[\Omega^2]=1$, none of the three roots is very large:" Cases (ii) and (iii) above show that all values of λ are, in fact, $\mathrm{O}(\varepsilon^{-2/3})$. However, this is of no consequence in Ross' study.

Let the solutions of Eqs. (I.67) and (I.68) be expressed as

$$w_m=\bar{w}_m(x,\nu,\Omega^2)+\tilde{w}_m(\xi,\mu,\nu,\Omega^2;\varepsilon) \tag{7}$$

$$\psi_m=\bar{\psi}_m(x,\nu,\Omega^2)+\varepsilon\tilde{\psi}_m(\xi,\mu,\nu,\Omega^2;\varepsilon) \tag{8}$$

where $\mathcal{B}(0)\{\bar{w}_m,\bar{\psi}_m\}=0$, $\mathcal{D}\{\bar{w}_m,\bar{\psi}_m\}=0$ and $\xi=x/\sqrt{\varepsilon}$ is a "fast" variable. To make all equations to follow dimensionless, we replace any unknown, call it f, by Lf, where $L\triangleq|\bar{w}_m(x;\nu,\Omega^2)|>0$, and then divide by L. For free vibrations L is arbitrary; for forced vibrations L is determined by the magnitude of the external force. Clearly, $\tilde{w}_m$ and $\tilde{\psi}_m$ are boundary-layer variables.

If $\varepsilon=0$, the solutions of Eqs. (I.67) and (I.68) are

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Table 1 Limiting form of Eqs. (13)–(16). $\Omega = O(H/R)^s$, $h/R = O(H/R)^r$

Case		Equation (13)	Equation (14)	Equation (15)	Equation (16)
			$s=1$		
I	$r>4$	$\textcircled{3}\underline{\underline{\textcircled{4}}}$	$\textcircled{3}\underline{\textcircled{4}}$	$\textcircled{2}$	$\textcircled{2}\textcircled{3}$
II	$r=4$	$\underline{\underline{\textcircled{1}}}\textcircled{3}\underline{\underline{\textcircled{4}}}$	$\underline{\underline{\textcircled{1}}}\textcircled{3}\underline{\textcircled{4}}$	$\textcircled{2}$	$\textcircled{2}\textcircled{3}$
			$s=\frac{1}{2}r-1$		
III	$2<r<4$	$\underline{\underline{\textcircled{1}}}\textcircled{3}$	$\underline{\underline{\textcircled{1}}}\textcircled{3}$	$\textcircled{2}$	$\textcircled{2}\textcircled{3}$
			$s=0$		
IV	$r=2$	$\textcircled{1}\textcircled{2}\textcircled{3}$	$\textcircled{1}\textcircled{2}\textcircled{3}$	$\textcircled{2}\textcircled{3}$	$\textcircled{1}\textcircled{2}\textcircled{3}$
V	$4/3<r<2$	$\textcircled{1}\textcircled{2}$	$\textcircled{1}\textcircled{2}$	$\textcircled{2}\textcircled{3}$	$\textcircled{1}$
VI	$r=4/3$	$\textcircled{1}\textcircled{2}$	$\textcircled{1}\textcircled{2}$	$\textcircled{1}\textcircled{2}\textcircled{3}$	$\textcircled{1}$
VII	$r<4/3$	$\textcircled{1}\textcircled{2}$	$\textcircled{1}\textcircled{2}$	$\textcircled{1}$	$\textcircled{1}$

$$\bar{w}_m = \hat{w}_m^{\pm} P_{\bar{\sigma}}^m(x) \quad \text{and} \quad \bar{\psi}_m = \frac{1-\Omega^2}{1+(1+\nu)\Omega^2}\bar{w}_m(x) \triangleq \bar{C}(\nu,\Omega^2)\bar{w}_m(x) \tag{9}$$

where

$$\bar{\sigma} = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{[1+(1+\nu)\Omega^2][2-(1-\nu)\Omega^2]}{1-\Omega^2}} = 1 + (1+\nu)\Omega^2 + O(\Omega^4), \quad \Omega^2 \neq 1 \tag{10}$$

2 Simplification of the Governing Algebraic Equations

As we showed in Sec. 11.2 of Ref. [1], if $\varepsilon = 0$, the lowest natural frequencies of the beam-shell system are the flexural frequencies of the beam given by Eq. (I.103), where $m=0$ and $m=1$ correspond to rigid body motions. We now trace the change in the *homogeneous forms* of the governing equations (I.80)–(I.83) as ε increases from zero. To this end let

$$\delta \triangleq H/R, \quad \varepsilon = \delta^r, \quad g = \bar{g}\delta^{2-r}, \quad k = \bar{k}\delta^{2-r}, \quad 0 < r,$$

$$\Omega = \bar{\Omega}\delta^s, \quad 0 \le s \le 1, \quad \bar{g},\bar{k},\bar{\Omega} = O(1) \tag{11}$$

and assume for simplicity that $B=H$.

Before beginning a perturbation analysis, we make some further simplifications of our beam equations (I.80)–(I.83). First, we neglect the underlined terms, which the subsequent analysis would show to be negligible. Next, we use Eq. (8) and to set $[\![\psi_m]\!] = [\![\bar{\psi}_m]\!] + O(\varepsilon)$. Then, we set

$$\varepsilon^2 [\![\Delta_m w_m]\!] = \varepsilon [\![\tilde{w}_m'']\!] + O(\varepsilon^2) \quad \text{and}$$

$$\varepsilon^2 [\![(\Delta_m w_m)_{,\phi}]\!] = -\varepsilon^{1/2}[\![\tilde{w}_m''']\!] + O(\varepsilon^2) \tag{12}$$

Finally, we retain only the dominant terms in each equation, rearrange the order in Eq. (I.83) so that in all equations the shell contributions come first, and number each term for future reference. (See Table 1.) Altogether,

$$-[\underset{\textcircled{1}}{1} + \underset{\textcircled{2}}{(1+\nu)}\bar{\Omega}^2\delta^{2s}][\![\chi_m]\!]\delta^{r-2} + (1+\nu)\{\bar{g}\bar{\Omega}^2(U_m \underset{\textcircled{3}}{+} mW_m)\delta^{2s} - (1/12)\bar{k}m(\underset{\textcircled{4}}{m^2}-1)^2 W_m\delta^2\} = 0 \tag{13}$$

$$-(\underset{\textcircled{1}}{1} + \underset{\textcircled{2}}{(1+\nu)}\bar{\Omega}^2\delta^{2s})[\![\bar{\psi}_m]\!]\delta^{r-2} + (1+\nu)\{\bar{g}\bar{\Omega}^2\underset{\textcircled{3}}{V_m}\delta^{2s} - (1/12)\bar{k}(m^2 - 1)[(\underset{\textcircled{4}}{m^2}-1)V_m + \Lambda_m]\delta^2\} = 0 \tag{14}$$

$$\underset{\textcircled{1}}{\delta^{(3r/2)-2}}[\![\tilde{w}''']\!] + \bar{k}(mU_{\underset{\textcircled{2}}{m}} - W_m) + \bar{g}\bar{\Omega}^2\underset{\textcircled{3}}{W_m}\delta^{2s} = 0 \tag{15}$$

and

$$\underset{\textcircled{1}}{[\![\tilde{w}''']\!]}\delta^{2r-4} + \bar{k}(GJ/E)\underset{\textcircled{2}}{m}^2\Lambda_m + (1/12)\bar{k}[(m^2 \underset{\textcircled{3}}{-} 1)V_m + \Lambda_m] = 0 \tag{16}$$

As $\delta \to 0$, consider the dominant terms in Eqs. (13)–(16) for different ranges of the exponents r and s. Listing only the magnitude of each term and noting Eq. (I.79), we have

$$\text{Eq. (13)} \Rightarrow \{\underline{\underline{\delta^{r-2}}}, \underline{\underline{\delta^{r+2s-2}}}, \delta^{2s}, \underline{\underline{\delta^2}}\} = 0 \tag{17}$$

$$\text{Eq. (14)} \Rightarrow \{\underline{\underline{\delta^{r-2}}}, \underline{\underline{\delta^{r+2s-2}}}, \delta^{2s}, \underline{\delta^2}\} = 0 \tag{18}$$

$$\text{Eq. (15)} \Rightarrow \{\delta^{3r/2-2}, \delta^0, \delta^{2s}\} = 0 \tag{19}$$

$$\text{Eq. (16)} \Rightarrow \{\delta^{2r-4}, \delta^0, \delta^0\} = 0 \tag{20}$$

In each of these four equations, two or more terms must either have the same magnitude or the dominant term must vanish as $\delta \to 0$. Terms with an r in the exponent represent the effects of the shell on the beam. As explained shortly, the terms with a double underline in Eqs. (17) and (18) and in Table 1 vanish if $m=0$ or $m=1$; the term in Eq. (18) with a single underline vanishes if $m=1$.

In Table 1 we have considered seven ranges of r, running from $r>4$ (when the influence of the shell is negligible) to $r<4/3$ (when the influence of the beam is negligible). We list by number only those terms in our basic Eqs. (13)–(16) that remain as $\delta \to 0$; the error terms in the full equations may be inferred from Eqs. (17)–(20).

It is noteworthy that, although our basic equations comprise 14 terms, the greatest number of terms in our limit equations is 11, which occurs only in *Case IV*. However, because *the Class I equations*, Eqs. (13) and (15)*, uncouple from the Class II equations*, Eqs. (14) and (16)*, there are at most two coupled equations with six terms*. In the extreme case $r<4/3$, the limit equations imply that $\tilde{w}_m(\xi;\mu,\nu,\Omega^2,\varepsilon) \equiv 0$ so that we are back to the natural frequencies of a complete spherical shell, as discussed by Niordson in Ref. [3].

3 Rapidly Varying Solutions

From Eq. (I.66), $\varepsilon\Delta_m = d^2/d\xi^2 + O(\varepsilon)$. Hence, Eqs. (I.67) and (I.68) take the forms

$$\frac{d^4\tilde{w}_m}{d\xi^4} + (1-\nu)^{-1}\left(\frac{d^2\tilde{\psi}_m}{d\xi^2} + 2\tilde{w}_m\right) - \Omega^2\tilde{w}_m = \mathrm{O}(\varepsilon) \tag{21}$$

and

$$\frac{d^2\tilde{\psi}_m}{d\xi^2} + (1+\nu)\tilde{w}_m = \mathrm{O}(\varepsilon) \tag{22}$$

where we have assumed that $\Omega^2=\mathrm{O}(1)$. Substituting Eq. (22) into Eq. (21), we have

$$(d^4/d\xi^4 + 1 - \Omega^2)\tilde{w}_m = \mathrm{O}(\varepsilon) \tag{23}$$

3.1 Low Dimensionless Frequencies ($\Omega^2 \leq 1-|c|$). If $0<|x|<1$, the decaying solutions of Eq. (23) have the form

$$\tilde{w}_m = e^{\pm p\xi}(A_m^{\pm}\cos p\xi \pm B_m^{\pm}\sin p\xi) + \mathrm{O}(\varepsilon), \quad p \triangleq \sqrt[4]{(1-\Omega^2)/4} \tag{24}$$

where $A_m^{\pm}$ and $B_m^{\pm}$ are unknown constants, and the plus (+) and minus (−) signs go with the solution in the upper or lower hemispheres, respectively. Equation (22) has the decaying solution

$$\tilde{\psi}_m = \frac{1+\nu}{\sqrt{1-\Omega^2}}e^{\pm p\xi}(B_m^{\pm}\cos p\xi \mp A_m^{\pm}\sin p\xi) + \mathrm{O}(\varepsilon) \tag{25}$$

We now express the four unknown constants in our final Eqs. (13)–(16) in terms of the three constants, $[\![\chi_m]\!]$, $[\![A_m]\!]$, and $\langle A_m\rangle$, and a new unknown constant, C_m, where $[\![A_m]\!] \triangleq A_m^+ - A_m^-$ and $\langle A_m\rangle \triangleq A_m^+ + A_m^-$, etc. The relations

$$\{[\![\bar{w}_{m,\phi}]\!], \langle\bar{w}_{m,\phi}\rangle\}P(\bar{\sigma},m) = \{\langle\bar{w}_m\rangle, [\![\bar{w}_m]\!]\}S(\bar{\sigma},m) \tag{26}$$

and

$$\{[\![\bar{\chi}_{m,\phi}]\!], \langle\bar{\chi}_{m,\phi}\rangle\}P(\bar{\tau},m) = \{\langle\bar{\chi}_m\rangle, [\![\bar{\chi}_m]\!]\}S(\bar{\tau},m) \tag{27}$$

follow from Eqs. (I.72) and (I.73) and will be used in what follows.

Being free of the small parameter ε, the solution of Eq. (I.69) is given by Eq. (I.74), as before, but which we now write as $\bar{\chi}_m = \hat{\chi}_m^{\pm}P_{\tau}^m(\mp x)$, in keeping with the notation of this section, from Eq. $(\mathrm{I}.62)_1$, (8) and (27)

$$U_m = \tfrac{1}{2}m\langle\psi_m\rangle + \tfrac{1}{2}\langle\bar{\chi}_{m,\phi}\rangle = \tfrac{1}{2}m\langle\bar{\psi}_m\rangle + \tfrac{1}{2}[\![\bar{\chi}_m]\!]S(\bar{\tau},m)/P(\bar{\tau},m) + \mathrm{O}(\varepsilon) \tag{28}$$

But by Eq. (26) and because $\bar{\psi}_m = \bar{C}\bar{w}_m$ and $[\![v_m]\!] = [\![\psi_{m,\phi}]\!] + m[\![\bar{\chi}_m]\!] = 0$

$$\langle\bar{\psi}_m\rangle = [\![\bar{\psi}_{m,\phi}]\!]P(\bar{\sigma},m)/S(\bar{\sigma},m) = -m[\![\bar{\chi}_m]\!]P(\bar{\sigma},m)/S(\bar{\sigma},m) + \mathrm{O}(\varepsilon^{1/2}) \tag{29}$$

Thus

$$U_m = H(\bar{\tau},\bar{\sigma},m)[\![\bar{\chi}_m]\!] + \mathrm{O}(\varepsilon^{1/2}) \tag{30}$$

where

$$H(\bar{\tau},\bar{\sigma},m) \triangleq \frac{1}{2}\left[\frac{S(\bar{\tau},m)}{P(\bar{\tau},m)} - m^2\frac{P(\bar{\sigma},m)}{S(\bar{\sigma},m)}\right] \tag{31}$$

Even though $\bar{\sigma}$ may be complex for certain ranges of Ω^2, $P(\bar{\sigma},m)$, and $S(\bar{\sigma},m)$ *are always real*. This is proved in Appendix A because it is not immediately apparent from Eqs. (I.72) and (I.73), which contain gamma functions that, individually, are complex-valued if $\bar{\sigma}$ is.

For the Fourier component of the $\mathbf{e}_z$-component of the beam displacement, we have from Eqs. (I.62) and (27)

$$V_m = \tfrac{1}{2}\langle\psi_{m,\phi}\rangle + \tfrac{1}{2}m\langle\bar{\chi}_m\rangle = \tfrac{1}{2}\langle\bar{\psi}_{m,\phi}\rangle + \tfrac{1}{2}m[\![\bar{\chi}_{m,\phi}]\!]P(\bar{\tau},m)/S(\bar{\tau},m) + \mathrm{O}(\varepsilon^{1/2}) \tag{32}$$

By Eqs. $(9)_2$ and (26)

$$\langle\bar{\psi}_{m,\phi}\rangle = [\![\bar{\psi}_m]\!]S(\bar{\sigma},m)/P(\bar{\sigma},m) \tag{33}$$

On the other hand, $[\![u_m]\!] = m[\![\psi_m]\!] + [\![\bar{\chi}_{m,\phi}]\!] = 0$ so that, altogether, Eq. (32) takes the form

$$V_m = H(\bar{\sigma},\bar{\tau},m)[\![\bar{\psi}_m]\!] + \mathrm{O}(\varepsilon^{1/2}) \tag{34}$$

In Appendix B we show that $H(\bar{\sigma},\bar{\tau},0)$ and $H(\bar{\sigma},\bar{\tau},1)$ are $\mathrm{O}(\Omega^{-2})$, whereas

$$H(\bar{\sigma},\bar{\tau},m) \to \frac{2m^2-1}{2m(m^2-1)} \quad \text{as} \quad \Omega^2 \to 0 \quad \text{if} \quad m \geq 2 \tag{35}$$

Thus as $\Omega^2 \to 0$, Eqs. (30), (34), and (35) imply that

$$[\![\bar{\chi}_0, \bar{\chi}_1, \bar{\psi}_0, \bar{\psi}_1]\!] = \{U_0, U_1, V_0, V_1\}\mathrm{O}(\Omega^2) \tag{36}$$

The Fourier components of the $\mathbf{e}_r$-component of the beam deflection are given by

$$W_m = \frac{1}{2}\langle\bar{w}_m\rangle + \frac{1}{2}\langle\tilde{w}_m\rangle = \frac{1}{2}\left[\langle\tilde{w}_m\rangle - \frac{m[\![\bar{\chi}_m]\!]P(\bar{\sigma},m)}{\bar{C}S(\bar{\sigma},m)}\right] + \mathrm{O}(\varepsilon^{1/2}) \tag{37}$$

where we have used $[\![v_m]\!] = [\![\psi_{m,\phi}]\!] + m[\![\bar{\chi}_m]\!] = 0$ and Eq. (26) to set

$$\begin{aligned}\langle\bar{w}_m\rangle &= \bar{C}^{-1}\langle\bar{\psi}_m\rangle = \bar{C}^{-1}[\![\bar{\psi}_{m,\phi}]\!]P(\bar{\sigma},m)/S(\bar{\sigma},m)\\ &= -\bar{C}^{-1}m[\![\bar{\chi}_m]\!]P(\bar{\sigma},m)/S(\bar{\sigma},m) + \mathrm{O}(\varepsilon^{1/2})\end{aligned} \tag{38}$$

By Eq. (26) and $[\![w]\!] = [\![\bar{w}_m]\!] + [\![\tilde{w}_m]\!] = 0$, the Fourier components of the local rotation $\Lambda = w_{,\phi}(\theta, 1/2\pi)$ become

$$\Lambda_m = \tfrac{1}{2}\langle\bar{w}_{m,\phi}\rangle - \tfrac{1}{2}\varepsilon^{-1/2}\langle\tilde{w}_m'\rangle = -\tfrac{1}{2}[[\![\tilde{w}_m]\!]S(\bar{\sigma},m)/P(\bar{\sigma},m) + C_m] \tag{39}$$

where by Eq. (24)

$$\langle\tilde{w}_m'\rangle = p[\![A_m + B_m]\!] + \mathrm{O}(\varepsilon) \triangleq \varepsilon^{1/2}C_m \tag{40}$$

Also from Eq. (24) and because $[\![w_{m,\phi}]\!] = [\![\bar{w}_{m,\phi}]\!] - \varepsilon^{-1/2}[\![\tilde{w}_m']\!] = 0$

$$\begin{aligned}&[\![\tilde{w}_m]\!] = [\![A_m]\!] + \mathrm{O}(\varepsilon), \quad \langle\tilde{w}_m\rangle = \langle A_m\rangle + \mathrm{O}(\varepsilon),\\ &[\![\tilde{w}_m']\!] = p\langle A_m + B_m\rangle = \mathrm{O}(\varepsilon^{1/2})\end{aligned} \tag{41}$$

and

$$[\![\tilde{w}_m'']\!] = 2p^2[\![B_m]\!] + \mathrm{O}(\varepsilon), \quad [\![\tilde{w}''']\!] = 2p^3\langle B_m - A_m\rangle + \mathrm{O}(\varepsilon) \tag{42}$$

Noting Eqs. (40) and $(41)_2$, we have

$$[\![B_m]\!] = -[\![A_m]\!] + \mathrm{O}(\varepsilon^{1/2}) \quad \text{and} \quad \langle B_m\rangle = -\langle A_m\rangle + \mathrm{O}(\varepsilon^{1/2}) \tag{43}$$

Equations (41)–(43) allow all unknown constants appearing in the beam Eqs. (13)–(16) to be expressed in terms of the four unknown constants $[\![\bar{\chi}_m]\!]$, $[\![A_m]\!]$, $\langle A_m\rangle$, and C_m. However, because of the two classes of free vibrations discussed in Sec. 7 of Part I, *Eqs. (13) and (15) involve only the two unknown* $[\![\bar{\chi}_m]\!]$ *and* $\langle A_m\rangle$, *whereas Eqs. (14) and (16) involve only the two unknown* $[\![A_m]\!]$ *and* C_m, *so the values of the dimensional natural frequencies are determined by two second-order determinants*. There is one important proviso: the natural frequency must satisfy $\Omega^2 \leq 1-|c|$. We therefore now consider the alternative.

3.2 High Dimensionless Frequencies ($\Omega^2 \geq 1+|c|$). The situation here is a bit more subtle than when $\Omega^2 \leq 1-|c|$. (We assume that the intermediate case $\Omega^2 - 1 = \mathrm{O}(\varepsilon^{\alpha})$, $\alpha>0$ can be inferred by continuity.) Of the four solutions of Eq. (23), one is decaying, namely

$$\tilde{w}_m^{(2)} = \tilde{A}_m^{\pm}e^{\pm q\xi} + \mathrm{O}(\varepsilon), \quad q \triangleq \sqrt[4]{\Omega^2 - 1} > 0 \tag{44}$$

where the $\tilde{A}_m^{\pm}$ are unknown constants. Another solution grows as we move away from the equator and is discarded. Finally, there are two solutions that are purely oscillatory. These cannot be analyzed by the simple perturbation scheme used so far so we turn to the WKB method. (See Ref. [4] for a simple exposition.)

If Eqs. (I.67) and (I.68) are reduced to a single equation by eliminating either w_m or ψ_m, then the resulting sixth-order ordinary differential equation can be expressed as the product of three factors of the form $\Delta_m+\lambda_k$, where λ_k is given by Eq. (3) with $\alpha=0$. The solution corresponding to λ_1 is given by Eq. (I.74) and the solution corresponding to λ_2 by Eq. (44).

To construct the perturbation solution associated with $\lambda_3=\varepsilon^{-1}q^2+\mathrm{O}(1)$, we first make the change of variable $x=\tanh\,\eta$ to bring the equation $(\varepsilon\Delta_m+q^2)\tilde{w}_m^{(3)}=\mathrm{O}(\varepsilon)$ into the *normal form*

$$(\varepsilon d^2/d\eta^2+q^2\,\mathrm{sech}^2\,\eta)\tilde{w}_m^{(3)}=\mathrm{O}(\varepsilon) \tag{45}$$

Next, we seek solutions of the form

$$\tilde{w}_m^{(3)}=\exp[\varepsilon^{-1/2}g(\eta;\varepsilon^{1/2})]$$
$$g=g_0(\eta)+\varepsilon^{1/2}g_1(\eta)+\mathrm{O}(\varepsilon),\quad g(0;\varepsilon^{1/2})=0 \tag{46}$$

Substituting this expression into Eq. (45) and equating to zero the coefficients of ε^0 and $\varepsilon^{1/2}$, we have

$$g_0'^2+q^2\,\mathrm{sech}^2\,\eta=0\ \text{ and }\ g_0''+2g_0'g_1'=0 \tag{47}$$

Hence

$$g_0=\pm iq\int_0^\eta \mathrm{sech}\,\bar{\eta}d\bar{\eta}=\pm iq\tan^{-1}(\sinh\,\eta)\ \text{ and }\ g_1'=-\tfrac{1}{2}(\ln g_0')' \tag{48}$$

As we need only a particular solution of Eq. (48), we take $g_1=\ln\sqrt{\cosh\,\eta}$. Then, because $g_0,g_2,\ldots$ are imaginary and $g_1,g_3,\ldots$ are real, Eq. (46) takes the form

$$\tilde{w}_m^{(3)}=\sqrt{\cosh\,\eta}[1+\mathrm{O}(\varepsilon)]\exp\{\pm i\varepsilon^{-1/2}[q\beta(\eta)+\mathrm{O}(\varepsilon)]\},$$
$$\beta\triangleq\tan^{-1}(\sinh\,\eta) \tag{49}$$

But as $x\to\pm1$, $\eta\to\pm\infty$ and $\beta\to\pm1/2\pi$. Hence, to avoid blow-ups at $x=\pm1$, the solution for $\tilde{w}_m^{(3)}$ must be of the form

$$\tilde{w}_m^{(3)}(\eta)=\tilde{B}_m^{\pm}\sqrt{\cosh\,\eta}[1+\mathrm{O}(\varepsilon)]\sin\left[\varepsilon^{-1/2}q\left(\tfrac{1}{2}\pi\pm\beta\right)+\mathrm{O}(\varepsilon^{1/2})\right] \tag{50}$$

where, as before, the plus sign (+) is used in the upper hemisphere and the minus sign (−) in the lower hemisphere.

With $\tilde{w}_m=\tilde{w}_m^{(2)}+\tilde{w}_m^{(3)}$, we need the following results from Eqs. (44) and (50) to express basic equations (13)–(16) in terms of four unknown constants

$$[\![\tilde{w}_m]\!]=[\![\tilde{A}_m]\!]+[\![\tilde{B}_m]\!]\sin\,\gamma+\mathrm{O}(\varepsilon),\quad \gamma\triangleq\tfrac{1}{2}\varepsilon^{-1/2}q\pi+\mathrm{O}(\varepsilon^{1/2}) \tag{51}$$

$$\langle\tilde{w}_m\rangle=\langle\tilde{A}_m\rangle+\langle\tilde{B}_m\rangle\sin\,\gamma+\mathrm{O}(\varepsilon) \tag{52}$$

For first derivatives, we have

$$\left.\frac{d\tilde{w}_m^{(2)}}{d\xi}\right|_{\xi=0}=\pm q\tilde{A}_m^{\pm}+\mathrm{O}(\varepsilon) \tag{53}$$

$$\left.\frac{d\tilde{w}_m^{(3)}}{d\xi}\right|_{\xi=0}=\varepsilon^{1/2}\left.\frac{d\tilde{w}_m^{(3)}}{dx}\right|_{x=0}=\varepsilon^{1/2}\left.\frac{d\tilde{w}_m^{(3)}}{d\eta}\right|_{\eta=0}=\pm q\tilde{B}_m^{\pm}\cos\,\gamma+\mathrm{O}(\varepsilon) \tag{54}$$

and for second and third derivatives, with the aid of Eq. (45)

$$\left.\frac{d^2\tilde{w}_m^{(2)}}{d\xi^2}\right|_{\xi=0}=q^2\tilde{A}_m^{\pm}+\mathrm{O}(\varepsilon) \tag{55}$$

$$\left.\frac{d^2\tilde{w}_m^{(3)}}{d\xi^2}\right|_{\xi=0}=\varepsilon\left.\frac{d^2\tilde{w}_m^{(3)}}{dx^2}\right|_{x=0}=\varepsilon\left.\frac{d^2\tilde{w}_m^{(3)}}{d\eta^2}\right|_{\eta=0}=-q^2\tilde{w}_m^{(3)}(0)+\mathrm{O}(\varepsilon)$$
$$=-\tilde{B}_m^{\pm}q^2\sin\,\gamma+\mathrm{O}(\varepsilon) \tag{56}$$

$$\left.\frac{d^3\tilde{w}_m^{(2)}}{d\xi^3}\right|_{\xi=0}=\pm q^3\tilde{A}_m^{\pm}+\mathrm{O}(\varepsilon) \tag{57}$$

$$\left.\frac{d^3\tilde{w}_m^{(3)}}{d\xi^3}\right|_{\xi=0}=\varepsilon^{3/2}\left.\frac{d^3\tilde{w}_m^{(3)}}{dx^3}\right|_{x=0}=\varepsilon^{3/2}\left.\left(2\frac{d}{d\eta}+\frac{d^3}{d\eta^3}\right)\tilde{w}_m^{(3)}\right|_{\eta=0}$$
$$=-\varepsilon^{1/2}q^2\left.\frac{d\tilde{w}_m^{(3)}}{d\eta}\right|_{\eta=0}+\mathrm{O}(\varepsilon^{3/2})=\mp\tilde{B}_m^{\pm}q^3\cos\,\gamma+\mathrm{O}(\varepsilon) \tag{58}$$

Thus

$$[\![\tilde{w}_m'']\!]=q^2([\![\tilde{A}_m]\!]-[\![\tilde{B}_m]\!]\sin\,\gamma)+\mathrm{O}(\varepsilon) \tag{59}$$

and

$$[\![\tilde{w}_m^{1/2}]\!]=q^3(\langle\tilde{A}_m\rangle-\langle\tilde{B}_m\rangle\cos\,\gamma)+\mathrm{O}(\varepsilon) \tag{60}$$

We now express all unknown constants that appear in our basic equations (13)–(16) in terms of the four unknown constants $[\![\bar{\chi}_m]\!]$, $[\![\tilde{B}_m]\!]$, $\langle\tilde{B}_m\rangle$, and $\tilde{C}_m$, a new unknown constant. First, from Eqs. (53) and (54)

$$\langle\tilde{w}_m'\rangle=q([\![\tilde{A}_m]\!]+[\![\tilde{B}_m]\!]\cos\,\gamma)+\mathrm{O}(\varepsilon)\triangleq\varepsilon^{1/2}\tilde{C}_m \tag{61}$$

and

$$[\![\tilde{w}_m']\!]=q(\langle\tilde{A}_m\rangle+\langle\tilde{B}_m\rangle\cos\,\gamma)+\mathrm{O}(\varepsilon) \tag{62}$$

Then from Eq. (61)

$$[\![\tilde{A}_m]\!]=-[\![\tilde{B}_m]\!]\cos\,\gamma+\mathrm{O}(\varepsilon^{1/2}) \tag{63}$$

so that from Eq. (57)

$$[\![\tilde{w}_m'']\!]=-q^2[\![\tilde{B}_m]\!](\sin\,\gamma+\cos\,\gamma)+\mathrm{O}(\varepsilon^{1/2}) \tag{64}$$

Furthermore, because $[\![w_{m,\phi}]\!]=[\![\bar{w}_{m,\phi}]\!]-\varepsilon^{-1/2}[\![\tilde{w}_m']\!]=0$, Eq. (62) yields

$$\langle\tilde{A}_m\rangle=-\langle\tilde{B}_m\rangle\cos\,\gamma+\mathrm{O}(\varepsilon^{1/2}) \tag{65}$$

so that from Eq. (60)

$$[\![\tilde{w}''']\!]=-2q^3\langle\tilde{B}_m\rangle\cos\,\gamma+\mathrm{O}(\varepsilon^{1/2}) \tag{66}$$

Equations (30), (34), (37), and (39) for U_m, V_m, W_m, and Λ_m remain unchanged, except that the constant C_m in Eq. (39) must be replaced by the new constant $\tilde{C}_m$. To express these four equations in terms of the four unknown constants $[\![\bar{\chi}_m]\!]$, $[\![\tilde{B}_m]\!]$, $\langle\tilde{B}_m\rangle$, and $\tilde{C}_m$ we need only the expressions

$$[\![\tilde{w}_m]\!]=[\![\tilde{B}_m]\!](\sin\,\gamma-\cos\,\gamma)+\mathrm{O}(\varepsilon^{1/2})\ \text{ and }\ \langle\tilde{w}_m\rangle=\langle\tilde{B}_m\rangle(\sin\,\gamma-\cos\,\gamma)+\mathrm{O}(\varepsilon^{1/2}) \tag{67}$$

that follow from Eqs. (51), (52), (63), and (65).

4 Numerical Results

Reference [1] discusses Case I ($r>4$, $s=1$, $\beta=0$) of Table 1 (in which the effect of the shell is negligible), as well as the case $m=0$, when an exact torsional oscillation exists. See Eqs. (I.106) and (I.107).

Cases II and III involve *low* dimensionless frequencies, where $\Omega^2\to0$ as $H/R\to0$ and $H(\bar{\sigma},\bar{\tau},m)$ is given by Eqs. (B9) and (B10) in Appendix B. Thus, in Case II ($r=4$, $s=1$) there is only rigid body motion if $m=0$ or $m=1$, whereas if $m\geq2$, Eqs. (13), (15), and (30) yield

$$\bar{\Omega}^2 = \frac{m(m^2-1)}{m^2+1}\left[\frac{2}{(1+\nu)\bar{g}(2m^2-1)} + \frac{k}{g}\frac{m(m^2-1)}{12}\right] \quad (68)$$

Equations (14) and (16) in Case II can first be reduced to

$$-[\![\bar{\psi}_m]\!] + (1+\nu)[\bar{g}\bar{\Omega}^2 V_m + m^2(m^2-1)\bar{k}(GJ/E)\Lambda_m] = 0 \quad (69)$$

and

$$(GJ/E)m^2\Lambda_m + (1/12)[(m^2-1)V_m + \Lambda_m] = 0 \quad (70)$$

Using Eq. (34) to eliminate $[\![\bar{\psi}_m]\!]$ from Eq. (69) in favor of V_m and noting Eq. (36), we arrive at two linear homogeneous algebraic equations for V_m and Λ_m. If $m=0$ or $m=1$, there is rigid body motion only, whereas if $m\geq 2$, we have nontrivial solutions providing

$$\begin{vmatrix} 2m(m^2-1)(1-2m^2)^{-1} + (1+\nu)\bar{g}\bar{\Omega}^2 & (1+\nu)m^2(m^2-1)\bar{k}(GJ/E) \\ m^2-1 & 1+12m^2(GJ/E) \end{vmatrix} = 0 \quad (71)$$

That is, if

$$\bar{\Omega}^2 = m(m^2-1)\left[\frac{2}{(1+\nu)\bar{g}(2m^2-1)} + \frac{m(m^2-1)(k/g)(GJ/E)}{1+12m^2(GJ/E)}\right] \quad (72)$$

The equations for *Case III* $(2<r<4,\ s=1/2r-1)$ follow upon neglecting the doubly underlined terms in Eqs. (68) and (72).

In Cases IV–VI, $\bar{\Omega}=\Omega$ because $s=0$, and in Cases IV and V there is no need to distinguish between low and high dimensionless frequencies because $\tilde{w}_m$ never enters the picture.

If we use Eq. (15) in Case IV $(r=2,\ s=0)$ to eliminate W_m from Eq. (13) and then use Eqs. (30) and (68) to set $U_m = H(\bar{\tau},\bar{\sigma},m)[\![\bar{\chi}_m]\!]$, we obtain for the natural frequencies

$$[1-(g/k)\Omega^2][1+(1+\nu)\Omega^2] = (1+\nu)\bar{g}\Omega^2[1-(g/k)\Omega^2 + m^2]H(\bar{\tau},\bar{\sigma},m) \quad (73)$$

On the other hand if we use Eq. (34) to set $V_m = H(\bar{\sigma},\bar{\tau},m)[\![\bar{\psi}_m]\!]$, then Eq. (14) yields

$$1+(1+\nu)\Omega^2 = (1+\nu)\bar{g}\Omega^2 H(\bar{\sigma},\bar{\tau},m), \quad (74)$$

whereas Eq. (16) becomes, ultimately, merely an equation for C_m in terms of $[\![A_m]\!]$ or $\tilde{C}_m$ in terms of $[\![\tilde{B}_m]\!]$.

Although we list only a few typical solutions of Eqs. (73) and (74) in Tables 2 and 3 below, a *qualitative description* of the *infinite number* of possible solutions is useful for any future detailed computations. First, it is obvious from Eqs. (I.72) and (31) that there is an integer value $m\in[0,\bar{\sigma}]$ such that $|H(\bar{\sigma},m)|=\infty$ if $\bar{\sigma}=n=1,2,3,\ldots$. The corresponding values of Ω^2 follow from Eq. (10) as

$$\tilde{\Omega}_n^2 = \frac{1+3\nu \pm \sqrt{(3+\nu)^2 + (1-2\nu)(1-\nu)n(n+1) + n^2(n+1)^2}}{2(1-\nu^2)} \quad (75)$$

These are just the dimensionless frequencies for what Niordson [3] calls "vibrations of the second class" for a complete spherical shell. If we take the minus sign in Eq. (75), we can easily show that there is an infinite sequence of values, $0={}_-\tilde{\Omega}_1^2<{}_-\tilde{\Omega}_2^2<\cdots<1$ that approaches 1 from below; if we take the plus sign, there is an infinite sequence of values $1<{}_+\tilde{\Omega}_1^2<{}_+\tilde{\Omega}_2^2<\cdots$ that approaches ∞. For $n=1$, ${}_-\tilde{\Omega}_1^2=0$, ${}_+\tilde{\Omega}_1^2=3/(1-\nu)$. In Table 4 we reproduce the table in p. 322 of Ref. [3], modified to display values of ${}_-\tilde{\Omega}_2^2$ and ${}_+\tilde{\Omega}_2^2$ for $n=2$.

Furthermore, by Eqs. (I.73) and (31), it is obvious that there is an integer value $m\in[1,\bar{\tau}]$ such that $|H(\bar{\sigma},0)|=\infty$ if $\bar{\tau}=n=1,2,3,\ldots$. Values of the corresponding values of Ω^2 follow from Eq. (I.107) as

$$\tilde{\Omega}_n^2 = \frac{(n+2)(n-1)}{2(1+\nu)} \quad (76)$$

Consider Eq. (74) first. The graph of the right side has vertical asymptotes at $\Omega^2={}_-\tilde{\Omega}_n^2$ or ${}_+\tilde{\Omega}_n^2$. The graph of the left side of Eq. (74) is a straight line, and every intersection of the two graphs

Table 2 (a) Typical values of Ω^2 from Eq. (73) for $\nu=0$, $g=1$, and $g/k=1$ and (b) typical values of Ω^2 from Eq. (73) for $\nu=0.5$, $g=1$ and $g/k=1$

m	${}_-\Omega_1^2$	${}_-\Omega_2^2$	${}_+\Omega_1^2$	${}_+\Omega_1^2$
(a)				
0			3.2465	10.2662
1			1.2018	2.8651
2	0.5287	0.8890	2.4426	5.7397
(b)				
0			2.1644	6.8441
1	0.6953	0.8269	3.8934	10.2163
2	0.3897	0.8110	1.9415	5.8282

Table 3 (a) Typical values of Ω^2 from Eq. (74) for $\nu=0$ and $g=1$ and (b) typical values of Ω^2 from Eq. (74) for $\nu=0.5$ and $g=1$

m	${}_-\Omega_1^2$	${}_-\Omega_2^2$	${}_+\Omega_1^2$	${}_+\Omega_2^2$
(a)				
0	0.7700	0.9141	2.5404	8.4993
1	0.5548	0.8700	4.2384	5.2559
2	0.7102	0.8953	1.3267	7.3706
(b)				
0	0.6437	0.8492	5.0728	13.5727
1	0.4156	0.7793	3.2168	8.0635
2	0.5404	0.7984	5.5124	12.1701

Table 4 For $n=2$, smallest values of Ω^2 from Eq. (75) that are $\leqslant 1$

	$\nu=0.0$	$\nu=0.1$	$\nu=0.2$	$\nu=0.3$	$\nu=0.4$	$\nu=0.5$
${}_-\tilde{\Omega}_2^2$	0.6277	0.5962	0.5669	0.5399	0.5150	0.4920
${}_+\tilde{\Omega}_2^2$	6.3723	6.7776	7.3498	8.1414	9.2469	10.8413

Table 5 Solutions of Eq. (80) for a few values of ρ_S/ρ, $k/g=\rho_S E/\rho E_S$, and ν

	$\nu=0.0$			$\nu=0.5$		
ρ_S/ρ	$k/g=0.1$	$k/g=0.5$	$k/g=0.9$	$k/g=0.1$	$k/g=0.5$	$k/g=0.9$
0.5	0.9094	0.9572	0.9948	0.8931	0.9498	0.9938
1.0	0.9610	0.9820	0.9978	0.9532	0.9784	0.9974
2.0	0.9839	0.9926	0.9991	0.9806	0.9911	0.9990

corresponds to a dimensionless natural frequency of the beam-shell configuration. These are interwoven with the natural frequencies of the closed shell.

Now consider Eq. (73) for $\Omega^2 \neq k/g$. If we divide both sides by $1-(g/k)\Omega^2$, then, qualitatively, we have the same situation as we had with Eq. (74) except that $\bar{\sigma}$ and $\bar{\tau}$ are reversed.

In Case V $(4/3<r<2,\ s=0)$, Eq. (13) yields $[\![\bar{\chi}_m]\!]=0$ and hence, by Eq. (30), $U_m=0$ so that Eqs. (I.79) and (15) yield the Class I natural frequency

$$\Omega^2 = k/g = E\rho_S/E_S\rho \tag{77}$$

Class II natural frequencies are those of a closed spherical shell, as discussed by Niordson [3].

In Case VI $(r=4/3,\ s=0)$, Eq. (13) again yields $[\![\bar{\chi}_m]\!]=U_m=0$, whereas Eq. (15) reduces to

$$[\![\tilde{w}''']\!] + (\bar{g}\Omega^2 - \bar{k})W_m = 0 \tag{78}$$

where by Eq. (37), $W_m=1/2\langle\tilde{w}_m\rangle$. Equations (14) and (16) imply, again that the Class II natural frequencies in this case are those of a closed spherical shell.

For *low frequency* vibrations $(\Omega^2 \leq 1-|c|)$, Eq. (78) becomes by Eqs. (41)–(43)

$$\bar{g}\Omega^2 = \bar{k} + 8p^3 \tag{79}$$

To further reduce this equation, recall that in Part II, $B=H$ (a beam of square cross section) so that by Eqs. (I.79) and (24): (i) $g \triangleq (\rho/\rho_S)(H^2/hR)=\bar{g}(H/R)^{2/3}$; (ii) $\bar{k}/\bar{g}=E\rho_S/E_S\rho$; and (iii) $4p^4=1-\Omega^2$. Hence

$$\begin{aligned}\Omega^2 &= \frac{\rho_S}{\rho}\left[\frac{E}{E_S} + \frac{2\sqrt{2}hR^{1/3}(1-\Omega^2)^{3/4}}{H^{4/3}}\right]\\ &= \frac{\rho_S}{\rho}\left[\frac{E}{E_S} + 4\sqrt{6(1-\nu^2)}(1-\Omega^2)^{3/4}\right]\end{aligned} \tag{80}$$

because from Eq. $(\mathrm{I.21})_1$, $\varepsilon^2=h^2/12(1-\nu^2)R^2=(H/R)^{8/3}$ in Case VI. There are no solutions to Eq. (80) if $\rho_S E/\rho E>1$. Although Eq. (80) is a quartic in $(1-\Omega^2)^{1/4}$, the exact solution is complicated and unenlightening. It is simpler to use a standard numerical routine (as we have) to determine values of Ω^2 for specific values of the three dimensionless parameters $k/g=\rho_S E/\rho E_S$, ρ_S/ρ, and ν. Thus, in Table 5, we list some typical solutions of Eq. (80).

For *high frequency* vibrations $(\Omega^2 \geq 1+|c|)$, we have, by Eqs. (66), $(67)_2$, and (78)

$$\bar{g}\Omega^2 = \bar{k} + 4q^3 \cos\gamma/(\sin\gamma - \cos\gamma) \tag{81}$$

or by Eq. (44)

$$\Omega^2 = \frac{\rho_S}{\rho}\left[\frac{E}{E_S} + 8\sqrt{3(1-\nu^2)}(\Omega^2-1)^{3/4}\frac{\cos\gamma}{\sin\left(\gamma-\frac{1}{4}\pi\right)}\right] \tag{82}$$

where from Eq. (51) and because $\varepsilon=(H/R)^{4/3}$ in Case VI

$$\gamma = \tfrac{1}{2}\pi(R/H)^{2/3}\sqrt[4]{\Omega^2-1} \tag{83}$$

Because $\cos\gamma/\sin(\gamma-1/4\pi)$ is π-periodic and can assume any real value, Eq. (82) has an infinite number of solutions such that $\Omega^2>1$. However, even though Eq. (82) is transcendental, a simple perturbation analysis allows us to compute explicitly many of the values of Ω^2 near 1, which occurs if the sine function in Eq. (82) is nearly 0.

Thus, let

$$\gamma = \left(\frac{1}{4}+n\right)\pi + \left(\frac{H}{R}\right)^2\kappa, \quad \kappa=\mathrm{O}(1), \quad n=0,1,2,\ldots \tag{84}$$

Then,

$$\frac{\cos\gamma}{\sin\left(\gamma-\frac{1}{4}\pi\right)} = \frac{1}{(H/R)^2\kappa}\left[1+\mathrm{O}\left(\frac{H}{R}\right)^2\right] \tag{85}$$

so that Eq. (83) yields

$$\Omega^2 = 1 + \left(\frac{1+4n}{2}\right)^4\left(\frac{H}{R}\right)^{8/3}\left[1 + \frac{16}{1+4n}\left(\frac{H}{R}\right)^2\frac{\kappa}{\pi} + \mathrm{O}\left(\frac{H}{R}\right)^4\right] \tag{86}$$

The value of κ then follows from Eq. (82) as

$$\kappa = \frac{8\sqrt{3(1-\nu^2)}\left(\frac{1}{2}+2n\right)^3}{1-\rho_S E/\rho E_S} + \mathrm{O}\left(\frac{H}{R}\right)^2 \tag{87}$$

However, *the reader must take Eqs. (86) and (87) with a grain of salt* because, as Eq. (3) shows, our perturbation analysis in Secs. 3.2 and 3.3 is valid only if $\Omega^2-1=\mathrm{O}(\varepsilon^\alpha)$ with $0\leq\alpha<2/3$, a restriction that is clearly violated in Eq. (86). A correct determination of Ω^2 in the transition zone requires a consideration of the Eqs. (4) and (5) and would take us beyond the scope of this paper. For an enlightening *numerical analysis* of the natural frequencies of a spherical shell with a hole, where a similar transition zone occurs, see Ref. [5].

5 Conclusion

To simplify considerably the "exact" equations developed in Part I, we used perturbation methods to exploit the small values of h/R and H/R, where h and H are, respectively, the thicknesses of the shell and the beam (of square cross section) and R is the midsurface radius of the shell. Thus, in many cases, explicit formulas for the square of the dimensionless natural frequency Ω^2 emerge for certain ranges of the ratio h/H, whereas other ranges of this ratio lead to relatively simple scalar equations that may be readily solved numerically by standard techniques. Typical solutions of these latter frequency equations are given. The rather elaborate behavior of solutions in the transition zone, $|\Omega^2-1|=\mathrm{O}(\varepsilon^{2\alpha/3})$, $\alpha>0$, remains to be carried out.

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Appendix A: Proof That $P(\bar{\sigma},m)$ and $S(\bar{\sigma},m)$ are Real Valued

The reciprocal of the gamma function is an entire function of its argument and may be represented by the infinite product, Ref. [6], p. 1

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n} \tag{A1}$$

where γ is Euler's constant. Because the infinite product converges uniformly for all bounded arguments, we have

$$\frac{1}{\Gamma(z)\Gamma(\zeta)} = z\zeta e^{\gamma(z+\zeta)}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)\left(1+\frac{\zeta}{n}\right)e^{-(z+\zeta)/n} \tag{A2}$$

with $z=1+(1/2)\bar{\sigma}-(1/2)m$ and $\zeta=1/2-(1/2)\sigma-(1/2)m$, we find that

$$z\zeta = (m-1)(m-2)-\bar{\lambda} \tag{A3}$$

where

$$\left(1+\frac{z}{n}\right)\left(1+\frac{\zeta}{n}\right) = 1+\frac{1}{n}\left(\frac{3}{2}-m\right)+\frac{1}{n^2}[(m-1)(m-2)-\bar{\lambda}] \tag{A4}$$

and from Eq. (10)

$$\bar{\lambda} = \bar{\sigma}(\bar{\sigma}+1) = \frac{[1+(1+\nu)\Omega^2][2-(1-\nu)\Omega^2]}{1-\Omega^2}, \quad \Omega^2 \neq 1 \tag{A5}$$

This last equation shows that the right side of Eq. (I.72) is a function of $\bar{\lambda}$ and m only and thus real because $\bar{\lambda}$ is real. A nearly identical argument shows that the right side of Eq. (I.73) is likewise a function of $\bar{\lambda}$ and m only.

Appendix B: Values of the Function $H(\bar{\sigma},\bar{\tau},m)$ When Ω^2 is Small

If $\sigma=1$

$$P(1,m) = \begin{cases} 0 & \text{if } m=0 \\ -1 & \text{if } m=1 \\ 0 & \text{if } m\geq 2 \end{cases} \quad \text{and} \quad S(1,m) = \begin{cases} 1 & \text{if } m=0 \\ 0 & \text{if } m\geq 1 \end{cases} \tag{B1}$$

Eqs. (I.107) and (10) show that $\bar{\sigma},\bar{\tau}=1+O(\Omega^2)$. Hence, as $\Omega^2 \to 0$, the function $H(\bar{\sigma},\bar{\tau},m)$ defined in Eq. (31) becomes infinite implying that $[\![\bar{\chi}_m]\!]\to 0$ as $\Omega^2\to 0$. To see precisely how $[\![\bar{\chi}_m]\!]$ goes to zero with Ω^2, let $\bar{\sigma}=1+\kappa\Omega^2$, where $\kappa=1+\nu$ or $\kappa=(2/3)\times(1+\nu)$ according to Eq. (10) or Eq. (I.107), and consider the two cases $m=2n$ and $m=2n+1$, $n=0,1,2,\ldots$.

If $m=2n$, $n\geq 0$ in Eq. (I.72)

$$\cos\left[\tfrac{1}{2}\pi(1+2n+\kappa\Omega^2)\right] \sim (-1)^{n+1}(\kappa\pi/2)\Omega^2 \quad \text{as} \quad \Omega^2 \to 0 \tag{B2}$$

so that the formulas on pp. 2 and 3 of Ref. [6] for the gamma function yield

$$P(1+\kappa\Omega^2,2n) \sim \frac{(2n)!}{2n-1}\kappa\Omega^2 \quad \text{as} \quad \Omega^2\to 0 \tag{B3}$$

If $m=2n$, $n\geq 1$

$$\frac{1}{\Gamma\left(1+\frac{1}{2}\kappa\Omega^2-n\right)} \sim \tfrac{1}{2}(-1)^{n-1}(n-1)!\,\kappa\Omega^2 \quad \text{as} \quad \Omega^2\to 0 \tag{B4}$$

so that

$$S(1+\kappa\Omega^2,2n) \sim -(2n+1)(2n-1)!\,\kappa\Omega^2 \quad \text{as} \quad \Omega^2\to 0 \tag{B5}$$

If $m=2n+1$, $n\geq 0$

$$\sin\left[\pi(n+1)+\tfrac{1}{2}\pi\kappa\Omega^2\right] \sim (-1)^{n+1}(\kappa\pi/2)\Omega^2 \quad \text{as} \quad \Omega^2\to 0 \tag{B6}$$

so that from Eq. (I.73)

$$S(1+\kappa\Omega^2,2n+1) \sim -2(n+1)(2n)!\,\kappa\Omega^2 \quad \text{as} \quad \Omega^2\to 0 \tag{B7}$$

whereas if $m=2n+1$, $n\geq 1$, Eqs. (I.72) and (B4) yield

$$P(1+\kappa\Omega^2,2n+1) \sim (2n+1)(2n-1)!\,\kappa\Omega^2 \quad \text{as} \quad \Omega^2\to 0 \tag{B8}$$

Thus, in summary, as $\Omega^2\to 0$

$$H(\bar{\sigma},\bar{\tau},0) \sim -\frac{1}{2(1+\nu)\Omega^2}, \quad H(\bar{\sigma},\bar{\tau},1) \sim -\frac{3}{8(1+\nu)\Omega^2} \tag{B9}$$

and

$$H(\bar{\sigma},\bar{\tau},m) \sim \frac{2m^2-1}{2m(m^2-1)}, \quad m\geq 2 \tag{B10}$$

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